

# 爱因斯坦场方程研究

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**摘 要:** 我们将介绍我们的工作: ①把9个关联系数分成两组: 涉及 $\ddot{r}$ (线加速度)的A组包括 $\Gamma_{01}^0, \Gamma_{00}^1, \Gamma_{11}^1, \Gamma_{22}^1$ , 和 $\Gamma_{33}^1$ ; 不涉及 $\ddot{r}$ 的B组包括 $\Gamma_{12}^2, \Gamma_{33}^2, \Gamma_{13}^3$ , 和 $\Gamma_{23}^3$ 。②回顾关系式 $(\ddot{r})_r = (\ddot{r})^N \gamma^3$ , 这里 $(\ddot{r})_r$ 是相对论力学中的线加速度,  $(\ddot{r})^N$ 是牛顿力学中的线加速度,  $\gamma$ 是洛伦兹因数。③我们已经确定 $(\Gamma_{01}^0)^N = -b \dot{b}^{-1}, (\Gamma_{00}^1)^N = -c^2 b \dot{b}^{-5}, (\Gamma_{11}^1)^N = b \dot{b}^{-1}, (\Gamma_{22}^1)^N = -rb^{-2}, (\Gamma_{33}^1)^N = -rb^{-2} \sin^2 \theta$ 。④我们已经确定 $(\Gamma_{01}^0)_r = (-b \dot{b}^{-1}) \gamma^3, (\Gamma_{00}^1)_r = (-c^2 b \dot{b}^{-5}) \gamma^3, (\Gamma_{11}^1)_r = (b \dot{b}^{-1}) \gamma^3, (\Gamma_{22}^1)_r = (-rb^{-2}) \gamma^3, (\Gamma_{33}^1)_r = (-rb^{-2} \sin^2 \theta) \gamma^3$ 。⑤我们已经证明 $\Gamma_{01}^0 = A'/(2A) = -b \dot{b}^{-1}, \Gamma_{00}^1 = A'/(2B) = -c^2 b \dot{b}^{-5}, \Gamma_{11}^1 = b \dot{b}^{-1}, \Gamma_{22}^1 = -rb^{-1} = -rb^{-2}, \Gamma_{33}^1 = -rb^{-1} \sin^2 \theta = -rb^{-2} \sin^2 \theta$ 。⑥我们已经确定 $(\Gamma_{12}^2)_r = r^{-1}, (\Gamma_{33}^2)_r = \sin \theta \cos \theta, (\Gamma_{13}^3)_r = r^{-1}, (\Gamma_{23}^3)_r = \cot \theta$ 。⑦我们分析了Schwarzschild解并得出两个结论: (a)  $B = (1 - 2GM/(c^2 r))^{-1} = (1 - \dot{r}^2/c^2)^{-1} = \gamma^2$ , 这表明它和牛顿守恒定律有关。(b) 对于弱引力场 $GM/(c^2 r) \ll 1, B = (1 - 2GM/(c^2 r))^{-1} \approx 1 + 2GM/(c^2 r) \approx 1 + 2GM/(c^2 r) + (GM/(c^2 r))^2 = (1 + GM/(c^2 r))^2 = \gamma^2$ , 因此,  $\gamma = 1 + GM/(c^2 r) = b$ , 这个关系式对强引力场也适用。把这些需要的表达方式带入方程式, 并应用关系式, 我们能简化方程式。我们已经得到了相对论解:  $-c^2 d\tau^2 = c^2 (1 + GM/(c^2 r))^{-2} dt^2 - (1 + GM/(c^2 r))^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$ 。

**关键词:** 广义相对论; 场方程; 引力理论

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## A Study of Einstein Field Equation

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**Abstract:** In this paper we present our research work: ①We divide 9 connection coefficients into two groups: Group A involving  $\ddot{r}$  (the linear acceleration) includes  $\Gamma_{01}^0, \Gamma_{00}^1, \Gamma_{11}^1, \Gamma_{22}^1$ , and  $\Gamma_{33}^1$ ; Group B not involving  $\ddot{r}$  includes  $\Gamma_{12}^2, \Gamma_{33}^2, \Gamma_{13}^3$ , and  $\Gamma_{23}^3$ . ②Recalling the relation formula:  $(\ddot{r})_r = (\ddot{r})^N \gamma^3$ , where  $(\ddot{r})_r$  is the linear acceleration in relativistic mechanics,  $(\ddot{r})^N$  the linear acceleration in Newtonian mechanics, and  $\gamma$  Lorentz factor. ③We have determined that  $(\Gamma_{01}^0)^N = -b \dot{b}^{-1}, (\Gamma_{00}^1)^N = -c^2 b \dot{b}^{-5}, (\Gamma_{11}^1)^N = b \dot{b}^{-1}, (\Gamma_{22}^1)^N = -rb^{-2}, (\Gamma_{33}^1)^N = -rb^{-2} \sin^2 \theta$ . ④We have determined that  $(\Gamma_{01}^0)_r = (-b \dot{b}^{-1}) \gamma^3, (\Gamma_{00}^1)_r = (-c^2 b \dot{b}^{-5}) \gamma^3, (\Gamma_{11}^1)_r = (b \dot{b}^{-1}) \gamma^3, (\Gamma_{22}^1)_r = (-rb^{-2}) \gamma^3, (\Gamma_{33}^1)_r = (-rb^{-2} \sin^2 \theta) \gamma^3$ . ⑤We have proved that  $\Gamma_{01}^0 = A'/(2A) = -b \dot{b}^{-1}, \Gamma_{00}^1 = A'/(2B) = -c^2 b \dot{b}^{-5}, \Gamma_{11}^1 = b \dot{b}^{-1}, \Gamma_{22}^1 = -rb^{-1} = -rb^{-2}, \Gamma_{33}^1 = -rb^{-1} \sin^2 \theta = -rb^{-2} \sin^2 \theta$ . ⑥We have determined that  $(\Gamma_{12}^2)_r = r^{-1}, (\Gamma_{33}^2)_r = \sin \theta \cos \theta, (\Gamma_{13}^3)_r = r^{-1}, (\Gamma_{23}^3)_r = \cot \theta$ . ⑦We have analyzed Schwarzschild solution, and drawn two conclusions: (a)  $B = (1 - 2GM/(c^2 r))^{-1} = (1 - \dot{r}^2/c^2)^{-1} = \gamma^2$ .

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$c^2)^{-1} = \gamma^2$ , indicating that it involves Newtonian formula of energy conservation. (b) In the case of the weak gravitational field,  $GM/(c^2 r) \ll 1$ ,  $B = (1 - 2GM/(c^2 r))^{-1} \approx 1 + 2GM/(c^2 r) \approx 1 + 2GM/(c^2 r) + (GM/(c^2 r))^2 = (1 + GM/(c^2 r))^2 = \gamma^2$ , therefore,  $\gamma = 1 + GM/(c^2 r) = b$ , it holds too, in the case of the strong gravitational field. (8) Substituting the needed expressions into equations, and applying the relation formulas, we can simplify the equations, Obtaining the relativistic solution:  $-c^2 d\tau^2 = c^2 (1 + GM/(c^2 r))^{-2} dt^2 - (1 + GM/(c^2 r))^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2$

**Keywords:** general relativity; field equation; gravitational theory

## 0 Introduction

Schwarzschild solution for Einstein field equation has made an important contribution to development of general relativity, but we should point out Schwarzschild solution belongs to Newtonian mechanics, and it is suitable only to the weak gravitational field, but not suitable to the strong gravitational field. In this paper we will give sufficient basis for this conclusion.

## 1 Classification of connection coefficients

We divide 9 connection coefficients into two groups: group A involving  $\ddot{r}$  (the linear acceleration), see Table 1; group B not involving  $\ddot{r}$ , see Table 1 and 2.

Table 1. Group A involving  $\ddot{r}$

$$\Gamma_{01}^0 ( = A'/(2A) = -B'/(2B) ) = -\Gamma_{11}^1 = \ddot{r}/(\dot{r}\dot{r}), \\ \Gamma_{00}^1 = -\ddot{r}/(\dot{t}\dot{t}), \Gamma_{11}^1 = -\ddot{r}/(\dot{r}\dot{r}), \Gamma_{22}^1 = -\ddot{r}/(\dot{\theta}\dot{\theta}), \Gamma_{33}^1 = -\ddot{r}/(\dot{\phi}\dot{\phi})$$

Table 2. Group B not involving  $\ddot{r}$

$$\Gamma_{12}^2 = -\ddot{\theta}/(\dot{r}\dot{\theta}), \Gamma_{33}^2 = -\ddot{\theta}/(\dot{\phi}\dot{\phi}), \Gamma_{13}^3 = -\ddot{\phi}/(\dot{r}\dot{\phi}), \Gamma_{23}^3 = -\ddot{\phi}/(\dot{\theta}\dot{\phi})$$

## 2 $\ddot{r}_R = \ddot{r}^N \gamma^3$

$\ddot{r}_R$  is linear acceleration in relativistic mechanics,  $\ddot{r}^N$  is linear acceleration in Newtonian mechanics, the relation formula between  $\ddot{r}_R$  and  $\ddot{r}^N$  is as below (references [2], P38)

$$\ddot{r}_R = \ddot{r}^N \gamma^3 \quad (1)$$

where  $\gamma$  is Lorentz factor. So that the expressions of group A in two kinds of mechanics are different.

## 3 The expressions of group A in Newtonian mechanics

Example 1: Determine the expression of  $(\Gamma_{11}^1)^N$ .

Solution: From  $-c^2 d\tau^2 = -b^2 dr^2$  (see line element Exp. (4))

$\dot{r} = cb^{-1}$ . From  $\ddot{r}^N + (\Gamma_{11}^1)^N (\dot{r}\dot{r}) = 0$ , we can get

$$(\Gamma_{11}^1)^N = -\frac{dr}{d\tau} \frac{d\dot{r}}{dr} / (\dot{r}(cb^{-1})) = -(\dot{r}(-cb^{-2}b')) / (\dot{r}(cb^{-1})) = b'b^{-1}.$$

Following Example 1, we can obtain Table 3.

Table 3. The expressions of Group A in Newtonian mechanics

$$(\Gamma_{01}^0)^N = -b'b^{-1}, (\Gamma_{00}^1)^N = -c^2 b'b^{-5}, (\Gamma_{11}^1)^N = b'b^{-1}, (\Gamma_{22}^1)^N = -rb^{-2}, (\Gamma_{33}^1)^N = -rb^{-2} \sin^2 \theta.$$

## 4 The expressions of group A in relativistic mechanics

Example 2: Determine the expression of  $(\Gamma_{11}^1)_R$ .

Solution: From  $(\ddot{r})_R + (\Gamma_{11}^1)_R (\dot{r}\dot{r}) = 0$  we can get

$$(\Gamma_{11}^1)_R = -(\ddot{r})_R / (\dot{r}\dot{r}) = \gamma^3 (-\ddot{r}^N / (\dot{r}\dot{r})) = (\Gamma_{11}^1)^N \gamma^3.$$

Following Example 2 we can obtain Table 4.

Table 4. The expressions of group A in relativistic mechanics

$$(\Gamma_{01}^0)_R = (-b'b^{-1}) \gamma^3, (\Gamma_{00}^1)_R = (-c^2 b'b^{-5}) \gamma^3, \\ (\Gamma_{11}^1)_R = (b'b^{-1}) \gamma^3, \\ (\Gamma_{22}^1)_R = (-rb^{-2}) \gamma^3, (\Gamma_{33}^1)_R = (-rb^{-2} \sin^2 \theta) \gamma^3$$

## 5 Group A used in Schwarzschild solution

Example 3. Prove  $\Gamma_{00}^1 = A'/(2B) = -c^2 b'b^{-5}$

Proof.

$$\begin{aligned} \Gamma_{00}^1 &= A'/(2B) = (c^2 B^{-1})'/(2B) = (c^2 b^{-2})'/(2b^2) \\ &= -2c^2 b^{-3} b'/(2b^2) = -c^2 b' b^{-5} \end{aligned}$$

Following Example 3 we can obtain Table 5.

Table 5. Equivalence

$$\begin{aligned} \Gamma_{01}^0 &= A'/(2A) = -b' b^{-1}, \quad \Gamma_{00}^1 = A'/(2B) = -c^2 b' b^{-5}, \\ \Gamma_{11}^1 &= B'/(2B) = b' b^{-1}, \quad \Gamma_{22}^1 = -rB^{-1} = -rb^{-2}, \\ \Gamma_{33}^1 &= -rB^{-1} \sin^2 \theta = -rb^{-2} \sin^2 \theta \end{aligned}$$

## 6 The expression of Group B

The expressions of group B in two kinds of mechanics have no difference. See Table 6.

Table 6. The expression of group B

$$\begin{aligned} (\Gamma_{12}^2)_R^N &= r^{-1}, \quad (\Gamma_{33}^2)_R^N = \sin \theta \cos \theta, \quad (\Gamma_{13}^3)_R^N = r^{-1}, \\ (\Gamma_{23}^3)_R^N &= \cot \theta \end{aligned}$$

## 7 Analyzing Schwarzschild Solution

Schwarzschild solution is a very important result in general relativity, practice has proved that it is suitable for the weak gravitational field very well.

As a part of Schwarzschild solution  $B = (1 - 2GM/(c^2 r))^{-1}$  is related closely with Lorentz factor,

$$\gamma^2 = (1 - \dot{r}^2/c^2)^{-1} = (1 - 2GM/(c^2 r))^{-1} = B = b^2$$

It is very clear that Schwarz Schild solution involves a Newtonian formula of energy conservation.

$$\dot{r}^2/2 = GM/r \quad (2)$$

In the case of the weak gravitational field,  $GM/(c^2 r) < 1$ ,

$$\begin{aligned} b^2 = B &= (1 - 2GM/(c^2 r))^{-1} \approx 1 + 2GM/(c^2 r) \approx \\ &1 + 2GM/(c^2 r) + (GM/(c^2 r))^2 = (1 + GM/(c^2 r))^2 \\ &= \gamma^2 \end{aligned}$$

That is

$$\gamma = 1 + GM/(c^2 r) = b \quad (3)$$

Form. (3) holds still in the case of the strong gravitational field.

## 8 Solving Einstein field equation

Line element is expressed as below:

$$-c^2 d\tau^2 = a^2(r) dt^2 - b^2(r) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad (4)$$

In the following we will write  $a(r)$  and  $b(r)$  as  $a$  and  $b$ . The relation formulas between  $a$  and  $b$  are as following:

$$\begin{aligned} ab &= c \\ a'b + ab' &= 0 \end{aligned} \quad (5)$$

Einstein field equation can be written as below:

$$R_{\mu\nu} = \partial_\nu \Gamma_{\mu\sigma}^\sigma - \partial_\sigma \Gamma_{\mu\nu}^\sigma + \Gamma_{\mu\sigma}^\rho \Gamma_{\rho\nu}^\sigma - \Gamma_{\mu\nu}^\rho \Gamma_{\rho\sigma}^\sigma = 0 \quad (6)$$

Eq. (6) can be rewritten as following:

$$\begin{aligned} R_{00} &= -\partial_r \Gamma_{00}^1 + \Gamma_{00}^1 (\Gamma_{01}^0 - \Gamma_{11}^1) - \Gamma_{00}^1 (\Gamma_{12}^2 + \Gamma_{13}^3) = 0 \\ R_{11} &= \partial_r \Gamma_{01}^0 + \Gamma_{01}^0 (\Gamma_{01}^0 - \Gamma_{11}^1) - \Gamma_{11}^1 (\Gamma_{12}^2 + \Gamma_{13}^3) = 0 \\ R_{22} &= -\partial_r \Gamma_{22}^1 + \partial_\theta \Gamma_{23}^3 + \Gamma_{23}^3 \Gamma_{23}^3 = 0 \end{aligned} \quad (7)$$

Substituting the needed expressions of Tables 4 and 6 into Eq. (7), we obtain:

$$\begin{aligned} R_{00} &= c^2 b' b^{-5} \gamma^3 - c^2 b' b^{-5} \gamma^5 (5b' b^{-1} \gamma - 3\gamma' - 2b' b^{-1} \gamma) + 2c^2 b' b^{-5} \gamma^3 r^{-1} = 0 \\ R_{11} &= -b' b^{-1} \gamma^3 + b' b^{-1} \gamma^4 (b' b^{-1} \gamma^2 - 3\gamma' \gamma + 2b' b^{-1} \gamma^2) - 2b' b^{-1} \gamma^3 r^{-1} = 0 \\ R_{22} &= (rb^{-2} \gamma^3)' - 1 = 0 \end{aligned} \quad (8)$$

Applying  $\gamma = b$  (Form. 3), Eqs. (8) can be simplified as:

$$\begin{aligned} b'' + 2b' r^{-1} &= 0 \\ b'' + 2b' r^{-1} &= 0 \\ (rb)' - 1 &= 0. \end{aligned} \quad (9)$$

From Eq. (9) we can get

$$b = 1 + C/r \quad (10)$$

Comparing Exp. (10) with Form. (3) we can determine:

$$C = GM/c^2,$$

so that

$$b(r) = 1 + GM/(c^2 r) \quad (11)$$

using relation Form (5a) we can get

$$a(r) = c (1 + GM/(c^2 r))^{-1} \quad (12)$$

The relativistic solution can be expressed as

$$-c^2 d\tau^2 = c^2 (1 + GM/(c^2 r))^{-2} dt^2 - (1 + GM/(c^2 r))^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (13)$$

## Appendix A. Schwarzschild Solution

Line element can be written as below:

$$-c^2 d\tau^2 = A(r) dt^2 - B(r) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \quad (A1)$$

In the following we will write  $A(r)$  and  $B(r)$  as  $A$  and  $B$ . The relation formulas between  $A$  and  $B$  as following:

$$AB = c^2 \quad (\text{A2a})$$

$$A'B + Ab' = 0 \quad (\text{A2b})$$

The expressions of connection coefficients are as below:

$$\Gamma_{01}^0 = A'/(2A) = \frac{1}{2}A'A^{-1}, \Gamma_{00}^1 = A'/(2B) = \frac{1}{2}c^2A'A,$$

$$\Gamma_{11}^1 = B'/(2B) = -\frac{1}{2}A'A^{-1}, \Gamma_{22}^1 = -rB^{-1} = -$$

$$c^{-2}rA,$$

$$\Gamma_{33}^1 = -rB^{-1}\sin^2\theta = -c^2rA\sin^2\theta \quad (\text{A3a})$$

$$\Gamma_{12}^2 = r^{-1}, \Gamma_{33}^1 = \sin\theta\cos\theta, \Gamma_{13}^3 = r^{-1}, \Gamma_{23}^3 = \cot\theta \quad (\text{A3b})$$

Substituting the needed expressions in (A3a) and (A3b) into Eqs. (7), ones can obtain:

$$R_{00} = -\partial_r\left(\frac{1}{2}c^{-2}A'A\right) + \left(\frac{1}{2}c^{-2}A'A\right)\left(\frac{1}{2}A'A^{-1} - \left(-\frac{1}{2}A'A^{-1}\right)\right) - \left(\frac{1}{2}c^{-2}A'A\right)(r^{-1} + r^{-1}) = 0 \quad (\text{A4a})$$

$$R_{11} = \partial_r\left(\frac{1}{2}A'A^{-1}\right) + \left(\frac{1}{2}A'A^{-1}\right)\left(\frac{1}{2}A'A^{-1} - \left(-\frac{1}{2}A'A^{-1}\right)\right) - \left(-\frac{1}{2}A'A^{-1}\right)(r^{-1} + r^{-1}) = 0 \quad (\text{A4b})$$

$$R_{22} = -\partial_r(-rA/c^2) + \partial_\theta\cot\theta + \cot\theta\cot\theta = 0 \quad (\text{A4c})$$

Eqs. (A4a) ~ (A4c) can be simplified as

$$A'' + 2A'r^{-1} = 0 \quad (\text{A5a})$$

$$A'' + 2A'r^{-1} = 0 \quad (\text{A5b})$$

$$(rA)' / c^2 - 1 = 0 \quad (\text{A5c})$$

From Eq. (A5c) ones can get

$$A = c^2(1 + C/r).$$

Analyzing the gravitational field, a conclusion is given ([1], p. 121)

$$h_{00} = -2GM/(c^2r) = C/r,$$

So that

$$C = -2GM/c^2,$$

$$A(r) = c^2(1 - 2GM/(c^2r)) \quad (\text{A6a})$$

Applying relation Form. (A2a), ones can get

$$B(r) = (1 - 2GM/(c^2r))^{-1} \quad (\text{A6b})$$

Therefore Schwarzschild solution can be written as

$$-c^2 d\tau^2 = c^2 \left(1 - 2GM/(c^2r)\right) dt^2 - \left(1 - 2GM/(c^2r)\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \quad (\text{A7})$$

#### References:

- [1] FOSTER J., Nightingale J. D. A short Course in General Relativity [M]. Second Edition. 1995. New York: Springer Verlag, Chaps 2, 3.
- [2] RINDLER W. Introduction to Special Relativity [M]. Oxford: Clarendon press, 1982.

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